FILTER BANKS FOR POWER SPECTRUM ESTIMATION WITH A LOGARITHMICALLY UNIFORM FREQUENCY RESOLUTION

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LOGARITHMICALLY UNIFORM FREQUENCY RESOLUTION

by

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Abstract

Characteristics of different analog bandpass filter banks generating power spectra with a logarithmically uniform frequency resolution and a constant relative analysis bandwidth are discussed. Equivalent digital filters are derived for use with digital spectrum analyzers. As an example of application, an algorithm for digital third octave band analysis is demonstrated and used for evaluation of fluctuation power spectra characterizing the real part of impedance of standard RC circuits.

1. Introduction

In the study of stochastic signals as e.g. noise from different sources, it is often necessary to determine the power spectrum over a broad frequency range, rather than in narrow intervals or at special discrete frequencies. In such cases the analysis may involve frequencies being by orders of magnitude apart from each other, and consequently, the spectrum must be represented by using a logarithmic frequency scale. As common examples involving such broad range (logarithmic) spectra, the analysis of acoustical signals (Miller 1982) or of membrane noise (deFelice 1981) may be mentioned. The use of a logarithmic scale (often in both frequency and amplitude) may facilitate the analysis of basic features and of the origin of the signal. This can be illustrated by considering the real component of the electric impedance of an electrochemical cell. When both the reactance and frequency are represented on a logarithmic scale, equivalent circuit elements (such as, e.g., one corresponding to a charge transfer process or another being equivalent with a diffusional impedance) are easy to recognize, and to characterize in a quantitative manner (Bard and Faulkner 1980). The reactance versus frequency relationship can be derived, e.g., from the power spectrum of the spontaneous thermal noise of the cell constituents on the basis of the formula (Bezegh and Janata 1987)

$$Z_{Re}(f) = U^2(f)/(4kTB) = S(f)/(4kT) \qquad (1)$$

where f is frequency, Z_{Re} is the real component of impedance of the cell, $U^2(f)$ is the power spectrum and $S(f)$ is the spectral density of the voltage fluctuations derived from the cell impedance, and B is (absolute) bandwidth of the spectrum analyzer at frequency f.

Certain analog spectrum analyzers are able to generate a logarithmically uniform frequency scan and such filter bandwidths which vary proportionally with the frequency. Thus, the relative bandwidth (bandwidth/frequency) is nearly constant over the entire analyzed range. There are accepted standards recommending sets of frequencies and bandwidths for such analyzers (Acoustical Society of America 1986). One of the most common version, the third octave band analysis, is characterized ideally by a bank of rectangular filters, the center frequency of each of them being located at the frequency $2^{1/3}$ times its lower neighbor, and having a nominal bandwidth of $(2^{1/3}-1)$ times the center frequency. This mode of operation can be simulated algorithmically so that any digital spectrum analyzer may perform finally a very similar task, even though its operation is always based on a linearly uniform frequency distribution and a constant bandwidth in any span. A bank of such numerical filters not only yields a spectrum with nearly logarithmic final frequency distribution, but also eliminates the large abrupt changes in bandwidth, occurring at the boundaries between the adjacent frequency spans.

Common solutions to this problem (Hewlett-Packard 1978, Carnal and Rochelle 1984) exhibit, however, certain drawbacks. In one case (Hewlett-Packard 1978) the synthesized numerical filters unnecessarily mimic the non-idealities of the corresponding analog ones in that their bandedges are rounded. Another inconvenience of this algorithm is that it is instrument-specific, a consequence of which is the limited accessible frequency range (about 16 - 20,000 Hz). The filter passbands are not symmetrical with respect to their center frequencies on a logarithmic scale, rather they are systematically shifted towards lower frequencies that is a characteristic feature of other approaches (Carnal and Rochelle 1984) as well.

In this work a more general treatment of, and a solution to the problem of power spectrum determination with logarithmically uniform frequency resolution and a constant relative bandwidth are described. The digital version of the derived continuous bandpass filter characteristics is also discussed. As an example, third octave fluctuation analysis of a model RC circuit is performed.

2. The problem to be solved

Digital spectrum analyzers determine spectra with uniform (equidistant) frequency distribution and a constant absolute bandwidth in any frequency span. The frequency increment between individual spectrum points (called bins), and the analysis bandwidth are proportional to the frequency span actually used. In order to transform such spectra to one with a logarithmically uniform frequency resolution, it is necessary to select individual spectrum bins (or averages of groups involving bins of a given number) from several partly overlapping spans so that the resultant frequency distribution is close to a logarithmically uniform one. The bandwidth of the analysis then changes abruptly at the span boundaries within the same final spectrum. The magnitude of this change depends on the change in span, and may be of one order of magnitude or even more at each boundary where switching from one span to another takes place. The change of the relative bandwidth (bandwidth/frequency) is then also abrupt and large. In addition, its value is not even constant within the interval corresponding to one span, as it jumps up at the first point, and then decays hyperbolically to the last point. This hyperbolic relationship appears as an exponential decay on a log frequency axis (Fig 1, curve a).

To avoid this problem, a set of digital filters must be derived, which are characterized by a logarithmically uniform distribution of their

center frequencies, and, in addition, by a constant relative (or logarithmic) bandwidth over the entire analyzed frequency range. First, the equivalent continuous (analog) filter set is considered.

3. Ideal characteristics of a bank of logarithmically distributed continuous bandpass filters

A bank of bandpass filters of logarithmically uniform frequency distribution and of a constant relative bandwidth is characterized by the equations

$$f_{i+1} = q \, f_i \tag{2}$$

$$\Delta f_i = f_{i,H} - f_{i,L} = b \, f_i \tag{3}$$

where Δf is (absolute) bandwidth, $q > 1$ and b are constants, subscript i refers to the i-th filter the center frequency of which is f_i. Subscripts H and L refer to the higher and lower stopband frequencies (bandedges). For octave, half octave and third octave band analyses (Acoustical Society of America 1986) $q = 2$, $2^{1/2}$ and $2^{1/3}$, respectively.

In order to avoid any loss of energy of the signal, the bandedges of adjacent passbands must coincide. In this case b cannot be independent of q, and hence, equation 3 must be replaced by

$$f_{i,H} = f_{i+1,L} = q \, f_{i,L} \qquad\qquad (4)$$

On the other hand, for any passband symmetry

$$\Delta f_i = (q-1) \, f_{i,L} \qquad\qquad (5)$$

If any passband is <u>symmetrical</u> with respect to its center frequency <u>on a linear frequency scale</u>, then

$$f_i - f_{i,L} = f_{i,H} - f_i = q \, f_{i,L} - f_i$$

Thus,

$$f_{i,L} = f_i \, 2/(q+1) \qquad \text{and} \qquad f_{i,H} = f_i \, 2q/(q+1) \qquad (6a)$$

and with equation 2

$$\Delta f_i = f_i \, 2(q-1)/(q+1) \qquad\qquad (6b)$$

The same <u>symmetry</u> prescibed <u>on a logarithmic scale</u> implies

$$\log f_i - \log f_{i,L} = \log f_{i,H} - \log f_i \qquad \text{or:} \qquad f_i \, / \, f_{i,L} = f_{i,H} \, / \, f_i$$

Finally, with eq 4

$$f_{i,L} = f_i \, /\sqrt{q} \qquad \text{and} \qquad f_{i,H} = f_i \, \sqrt{q} \qquad (7a)$$

and with equation 2

$$\Delta f_i = f_i \, (\sqrt{q} - 1/\sqrt{q}) \qquad\qquad (7b)$$

Thus, at a linear symmetry $b = 2(q-1)/(q+1)$ while in the case of a logarithmic passband symmetry $b = \sqrt{q} - 1/\sqrt{q}$.

Bandpass filters with characteristics, $H_i(f)$, transform the original power spectrum as follows:

$$S^{\wedge}(f_i) = A_i \int_{f_{i,L}}^{f_{i,H}} S(f) \, H_i(f) \, df \qquad (8)$$

where $S^{\wedge}$ means the estimated spectrum obtained with the filter bank and A_i is the inverse norm of the i-th filter characteristics (i.e. the inverse of the integral of $H_i(f)$ over the passband). The equivalent of $H_i(f)$ on a logarithmic scale is $H_i^*(u)$ where $u = \log(f)$, with the normalizing factor, A_i^*. For <u>rectangular filters with linearly symmetrical passbands,</u> e.g.,

$$H_i = 1 \qquad \text{and} \qquad A_i = 1 / \Delta f_i = (q+1) / 2(q-1) \, f_i \qquad (9)$$

The two discussed symmetries are illustrated in Fig 2 for octave band analysis, when, as a special case, $\Delta f_i = f_{i,L}$ at any symmetry. The asymmetrical position of the center frequency on the logarithmic axis at linearly symmetrical passbands (curves a2) corresponds to higher "densities" of spectrum points at higher frequencies, due to the logarithmic representation. This means that a filter, rectangular on a linear frequency scale, can be considered to exhibit an increasing

exponential cha_acter on a logarithmic scale, because the "density" of the original spectrum points, s, increases exponentially along the logarithmic frequency axis, u:

$$s = df \, / \, d \log(f) = f \ln(a) = \ln(a) \, a^u \qquad (10)$$

where a is the logarithm base (ln means natural logarithm). Thus, equations 6 with rectangular filters (equations 9) define, in fact, a bank of logarithmic bandpass filters of asymmetrical and curved "sawtooth" type characteristics, with

$$H_i^*(u) = a^{(u-u_i)} \quad \text{and} \quad A_i^* = (q+1) \ln(a) \, / \, 2(q-1) \qquad (11)$$

The spectrum values at the center frequencies are not modified by these filters: $H_i^*(u_i)=1$ ($u_i = \log(f_i)$, see Fig 2-a2). A_i^* is the inverse integral of $H_i^*(u)$ over the interval defined by equations 6a, and is independent of i because the logarithmic bandwidth is constant. For similar reasons, the shapes of the filter characteristics are also identical in any passband (Fig 2-a2).

In order to obtain filters with symmetrical center frequency positions and rectangular characteristics on a logarithmic scale, equations 7 must be fulfilled. However, a proper weighting is then necessary on the linear frequency scale, to compensate for the required asymmetry of the linear scale passband. The weight must be inversely proportional to the spectrum density on the logarithmic scale (equation 10):

$$H_i = k / f \ln(a)$$

As its value should be unity at the actual center frequency, $k = f_i \ln(a)$ and so,

$$H_i(f) = f_i / f \qquad \text{and} \qquad A_i = 1 / f_i \ln(q) \qquad (12)$$

The normalizing factor is the inverse integral of the weighting function over the interval as defined by equations 7a. The filter characteristics, understandably, do not depend on the base of the logarithm, a. For the sake of completeness, the corresponding filters on a logarithmic scale are given also, as follows:

$$H_i^*(u) = 1 \qquad \text{and} \qquad A_i^* = 1 / \log(q) \qquad (13)$$

For both types of the derived filters (equations 9 and 12), and also for the corresponding logarithmic ones (equations 11 and 13), the spectrum to the "left" of the center frequency have equal weight with the "right" side within the same passband. For the linear scale filters, e.g.,

$$\int_{f_{i,L}}^{f_i} H_i(f) \, df = \int_{f_i}^{f_{i,H}} H_i(f) \, df \qquad (14)$$

This type of requirement is trivially fulfilled for the filters of equations 9 and 13 because of symmetry considerations. For logarithmically symmetrical filters on the linear scale, e.g, after insertion of equations 7a and 12 and after integration

$$\ln(f_i) - \ln(f_i/\sqrt{q}) = \ln(f_i \sqrt{q}) - \ln(f_i)$$

which is an algebraic identity. The proof for the logarithmical version of linearly symmetrical filters (equations 11) can be done similarly.

It is often easier to realize exponential filter characteristics than hyperbolic ones. At relatively narrow passbands, the filters of equation 12 can be approximated properly by

$$H_i(f) = \exp[-w_i (f-f_i)]$$

with $w>0$ constants. By inserting this equation into equation 14, and using the symmetry relationships (equations 7a)

$$1 - \exp[-w_i f_i (1/\sqrt{q}-1)] = \exp[-w_i f_i (\sqrt{q}-1)] - 1$$

and consequently,

$$x^{\sqrt{q}-1} + x^{1/\sqrt{q}-1} = 2 \tag{15}$$

where $x = \exp(-w_i f_i)$. Over the trivial solution to this equation ($x=1$, or $w_i=0$ that is $\underline{not}$ a solution to this problem) a constant x can be determined for $w_i > 0$ from equation 15 for any constant q in the interval $0<x<1$. Thus, $-w_i f_i = c$ is constant and independent of i, as it depends only on q. Consequently,

$$H_i = \exp[c(f/f_i - 1)] \tag{16a}$$

$$A_i = c /\{ \exp[c(\sqrt{q}-1)] - \exp[c(1/\sqrt{q}-1)] \} f_i \tag{16b}$$

The most important equations for both the linearly and logarithmically symmetrical filters are summarized in Table 1. For different selected values of q, the corresponding c constants, together with other data (bandwidth and stopband frequencies, the values of the weight function at the two stopbands, and the normalizing factors) are given in Table 2. As a consequence of equation 15, the "excess" of the exponential (approximate) weight function with respect to 1 at the lower bandedge is equal to its "shortage" at the higher stopband (see deviations from 1 in % in Table 2). This is not true for the theoretical (hyperbolic) characteristics. The value of c is approaching -1 as the analysis bandwidth decreases (at the limit when q=1 then c= -1). At third octave and half octave band analyses, the exponential filters approximate very closely the hyperbolic ones, as it is proven by the relative deviations of the two kinds of weight functions from 1 at the stopband frequencies in Table 2. At octave analysis the deviation of the approximate filter from the theoretical one is already more stressed, as it is seen in Fig 2-b1.

When no energy loss is allowed, then equations 2, 6, 8 and 9 define a filter set with logarithmic frequency distribution, constant relative (logarithmic) bandwidth and symmetrical passband on a linear scale, even though these filters are neither rectangular nor symmetrical on a logarithmic scale. In turn, the filters defined by equations 2, 7, 8 and 12 fulfill all these conditions, but they have curved "sawtooth" shape on a linear scale. So, both discussed filter banks have their

advantages and disadvantages. At narrow passbands, instead of equation 12, exponential filters (equations 16) can also be used.

An analogous treatment would be possible if energy loss were permitted (less compact filter bank) except that equation 3 should then be used in the derivations instead of equations 4, 5. Third octave band analysis as defined by the standard (Acoustical Society of America 1986) is only an approximation of a true energy-conserving scheme. At $q = 2^{1/3}$, according to Table 2, $\Delta f_i \approx 0.2300\ f_i$ at linearly symmetrical passbands and $\Delta f_i = 0.2316\ f_i$ in the case of a logarithmic symmetry, contrary to $\Delta f_i = (2^{1/3}-1)\ f_i = 0.2599\ f_i$ that is recommended by the standard (Acoustical Society of America 1986). A contradiction to equation 5 is easy to recognize here.

4. Equivalent digital filters

To derive equivalent digital filters, the lower and higher stopband frequencies belonging to each center frequency in question must be calculated (equations 6 or 7, respectively), and then, the integral in equation 8 must be computed. No weighting should be applied in the case of linearly symmetric passbands (equations 9), while the filters

according to equations 12 must be used in the logarithmically symmetrical case.

To select the bins of a given passband, half of the "original" instrument specific bandwidth, B, must also be taken into account at both bandedges when B is wide (when, e.g., using "flat top" shape window functions with the HP 3582A). The integration itself can be performed either by a simple summation for the bins belonging to the passband in question, or by more sophisticated integration schemes (Korn and Korn 1961). In the first case the normalizing factor, A_i, can be the inverse number of bins involved by the linearly symmetrical passband, while it is determined by summing all values of H_i for the given logarithmically symmetrical passband and inverting the sum. To reduce the errors of discretization, it is generaly better to compute the norms numerically instead of using equations 9 or 12.

The selection of the span from where the bins are used for calculation is quite trivial when the actual passband involves higher bins of the span, because then, many bins will add up to form one passband, providing a good approximation of the ideal (continuous) filter. However, at the lower end of any frequency span caution must be exercised because fewer bins are available there (as it is illustrated by the dotted lines in Fig 2-a2). This causes a poorer approximation because of the poorer digital resolution. This effect is even more stressed if the half-bandwidth inherent to the instrument, B, is also taken into account at

both band:dges when the bins for a given passband are selected. In addition, the DC part of the signal tends to bias and deform the low frequency end of any spectra in any span. For these reasons, it is better to switch to a lower span to reach lower frequencies rather than to use a very small number of the low bins of a higher span.

5. Testing the filters

5.1. Equipment used

The voltage fluctuations derived from the analyzed RC circuit were amplified by a gain of 1000 and then fed into an HP 3582A digital spectrum analyzer, that computed the raw spectra with linearly uniform frequency resolution. Its control via an HP-IB was performed by an HP-86B personal computer that computed the final spectrum with logarithmically uniform frequency resolution as well. Other experimental details are described elsewhere (Marecek *et al* 1988).

5.2. Digital third octave fluctuation analysis of standard RC circuits

The algorithm in (Hewlett-Packard 1978) used with the HP 3582A spectrum analyzer generates an approximate third octave band analysis with center frequencies recommended by the standard (Acoustical Society of America 1986) (Table 3, column 1). It is characterized by <u>linearly symmetrical</u> passbands, even though they do not correspond accurately to equations 6, and so, energy conservation is not strictly fulfilled (different passbands often overlap, see column 3). Thus, the algorithm realizes a bank of filters that are neither symmetrical nor rectangular on a logarithmic scale. In addition, the weighting functions as applied are not even rectangular on a linear scale (they are characterized by sloping bandedges), that means that such filters try to mimic even those typical non-idealities of the equivalent analog filters that are avoidable by digital processing. The span selection (column 2) in some cases enforces the use of very low bins (e.g. at 251.2 Hz) that is not the case of the filter bank realized with equations 7a and 12 in this work (columns 4, 5). These latter filters are rectangular and <u>symmetrical on a logarithmic scale</u>, fulfilling also energy conservation. The variance of the relative bandwidths is also smaller when using this algorithm, due to more favorable span selection (compare curves b and c in Fig 1). The determination of the stopbands and of the weighting function is fully algorithmic, and so, no restriction upon frequency range or instrument is imposed, contrary to the algorithm in (Hewlett-Packard 1978) that uses

numerically given weights and passband boundaries, which allows to access only the range of 16-20000 Hz.

As an example, fluctuation analysis of standard RC circuits have been performed. In order to apply correctly equation 1, first the original instrument bandwidths, B, of the different spans had to be used to determine properly scaled spectral densities, and, finally, reactances from the power spectra. Then, the transform to logarithmic scaling could be performed by the digital filters discussed. A result is shown in Fig 1, curve d which proves that, in spite of the extremely low energy of the fluctuation signal, already a few averages may provide a log reactance - log frequency plot, in which the scatter of the points is acceptable.

6. Conclusion

By using the equations given in this work, both the linearly and logarithmically symmetrical filters can be easily realized, with third octave, half octave, octave, or any other desired frequency resolution, without restrictions concerning frequency range or instrument, and whether an analog or digital spectrum analyzer has to be designed (though the digital version has many practical advantages). Both energy

conserving schemes and incontiguous filters with energy loss can be designed.

With an analogous treatment filters generating spectra with a logarithmically uniform wavenumber resolution can also be derived. Such spectra are not relevant for ordinary electrical noise measurements, but have significance e.g. in acoustical applications where the wavenumber representation is more commonly used.

Acknowledgement

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Table 1. Basic Equations for Logarithmically Uniform Filter Banks

	passband symmetry	
	linear	logarithmic
$f_{i,L}$	$f_i \; 2/(q+1)$	$f_i \; /\sqrt{q}$
$f_{i,H}$	$f_i \; 2q/(q+1)$	$f_i \; \sqrt{q}$
Δf_i	$f_i \; 2(q-1)/(q+1)$	$f_i \; (\sqrt{q} - 1/\sqrt{q})$
$H_i(f)$	1	$f_i \, / \, f$
A_i	$(q+1) \, / \, 2(q-1) \; f_i$	$1 \, / \, f_i \; \ln(q)$
$H_i^*(u)$	$a^{(u-u_i)}$	1
A_i^*	$(q+1) \; \ln(a) \, / \, 2(q-1)$	$1 \, / \, \log(q)$

1) ln means natural logarithm while log is the actual logarithmic function used for the definition of the logarithmic frequency axis (its base, a, may be 2, e, 10, or any other positive constant)

2) $u = \log(f)$

3) For exponential approximation of the logarithmically symmetrical filters see equations 16.

Table 2. Characteristics of Octave, Half Octave and Third Octave Band Analysis as Derived in This Work

name of analysis	q	symmetry of passband	Δf_i (*f_i)	$f_{i,L}$ (*f_i)	$f_{i,H}$ (*f_i)	c°	$d*$
-	5	lin.	1.3333	0.3333	1.6667	-	0.75
		log.	1.7889	0.4472	2.2361	-	0.6213
						+124, -55 %+	
		log., exponential approximation:				-0.9504	0.6876
						± 69 %	
octave	2	lin.	0.6667	0.6667	1.3333	-	1.5
		log.	0.7071	0.7071	1.4142	-	1.4427
						+41,-29 %	
		log., exponential approximation:				-0.9901	1.4716
						± 34 %	
1/2 octave $2^{1/2}$		lin.	0.3431	0.8284	1.1716	-	2.9142
		log.	0.3483	0.8409	1.1892	-	2.8854
						+19, -16 %	
		log., exponential approximation:				-0.9975	2.8998
						± 17 %	
1/3 octave $2^{1/3}$		lin.	0.2300	0.8850	1.1150	-	4.3473
		log.	0.2316	0.8909	1.1225	-	4.3281
						+12, -11 %	
		log., exponential approximation:				-0.9989	4.3377
						± 12 %	

° The values of c have been determined by solving numerically equation 15

* The factor d has been computed with the following definition:

$$A_i = d / f_i$$

+ Relative deviations of the weighting function at the two stopbands, $H(f_{i,L})$ and $H(f_{i,H})$, from unity. For octave analysis see also Fig 2b.

Table 3. Center Frequencies, Spans and Bins of Passbands of Third Octave Band Analysis

	nonalgorithmic as in (HP)		algorithmic (equations 7a)	
		symmetry		
	linear		logarithmic	
1	2	3	4	5
f_i (Hz)[a]	span (Hz)	bins[b]	span (Hz)	bins[b,c]
...			0 -	...
15.85		15 - 19	- 25	142 - 177
19.95		18 - 24		178 - 223
25.12		23 - 30		23 - 28
31.62		29 - 37		29 - 35
39.81	0 -	36 - 46		36 - 44
50.12	- 250	45 - 58		45 - 56
63.1		57 - 72	0 -	57 - 70
79.43		71 - 91	- 250	71 - 89
100		89 - 114		90 - 112
125.9		112 - 143		113 - 141
158.5		142 - 180		142 - 177
199.5		178 - 226		178 - 223
251.2		12 - 15		23 - 28
316.2		15 - 19		29 - 35
398.1		18 - 24		36 - 44
501.2		23 - 30		45 - 56
631		29 - 37	0 -	57 - 70
794.3	0 -	36 - 46	- 2500	71 - 89
1000	- 5000	45 - 58		90 - 112
1259		56 - 72		113 - 141
1585		71 - 91		142 - 177
1995		89 - 114		178 - 223
2512		112 - 143		23 - 28
3162		141 - 179		29 - 35
3981		178 - 225		36 - 44
5012		45 - 58		45 - 56

6310		57 - 72	0 -	57 - 70
7943	0 -	71 - 91	- 25000	71 - 89
10000	- 25000	89 - 114		90 - 112
12590		112 - 143		113 - 141
15850		142 - 179		142 - 177
19950		179 - 225		178 - 223
...				...

(HP): (Hewlett-Packard 1978)

[a] Nominal frequencies of the third octave band analysis, as defined in the standard (Acoustical Society of America 1986), being close, but not equal, to those corresponding to equation 2 (in average, the value of q is by about 0.1 % smaller according to the standard than its theoretical value of $2^{1/3}$).

[b] Bins belonging to the passband in question.

[c] Bins were selected with equations 7a and by rounding the obtained bin indices towards the actual center frequency. The original instrument bandwidth, B, was then added to calculate the bandwidths for curve c in Fig 1, to account for two half-bandwidths at both bandedges.

LEGENDS TO THE FIGURES

Figure 1

Relative bandwidth (bandwidth/frequency) versus log frequency (curves a-c) and result of fluctuation analysis (dots, d) with an HP 3582A spectrum analyzer. Any calculated bandwidth involves the frequency range covered by the bins belonging to the particular passband. In addition, in case c, 1/2 original instrument specific bandwidth, B, proper to the scan, at both "left" to the lowest and "right" to the highest bin of any passband, are added to the former value (B=0.006*SPAN for Hanning passband shape). In case b this has not been done, in order to account for the sloping bandedges. Ideal third octave relative bandwidths are marked as horizontal lines in a-c. Boundaries of adjacent frequency scans are shown by vertical lines.

(a) Relative bandwidths of moving average with 9 bin passbands.

(b) Relative bandwidths with linearly symmetrical third octave band analysis as in (Hewlett-Packard 1978).

(c) Relative bandwidths with logarithmically symmetrical third octave band analysis as derived in this work.

(d) Experimental result of fluctuation analysis of a parallel RC (R =1.8 GΩ, C =1 pF) with 4 averages.

Figure 2 Passband symmetries in case of an octave band analysis.

(a) Linearly symmetrical passbands on a linear (a1) and on a logarithmic (a2) frequency axis. Dots belong to frequencies spaced equidistantly on a linear axis, with increments of 1/20 times the first center frequency. Natural logarithm (ln) was used for the logarithmic representation.

(b) Logarithmically symmetrical passbands on a linear (b1) and on a logarithmic (b2) frequency axis.
hyp: calculated with the correct hyperbolic formula (equations 12)
exp: calculated with the approximate exponential formula
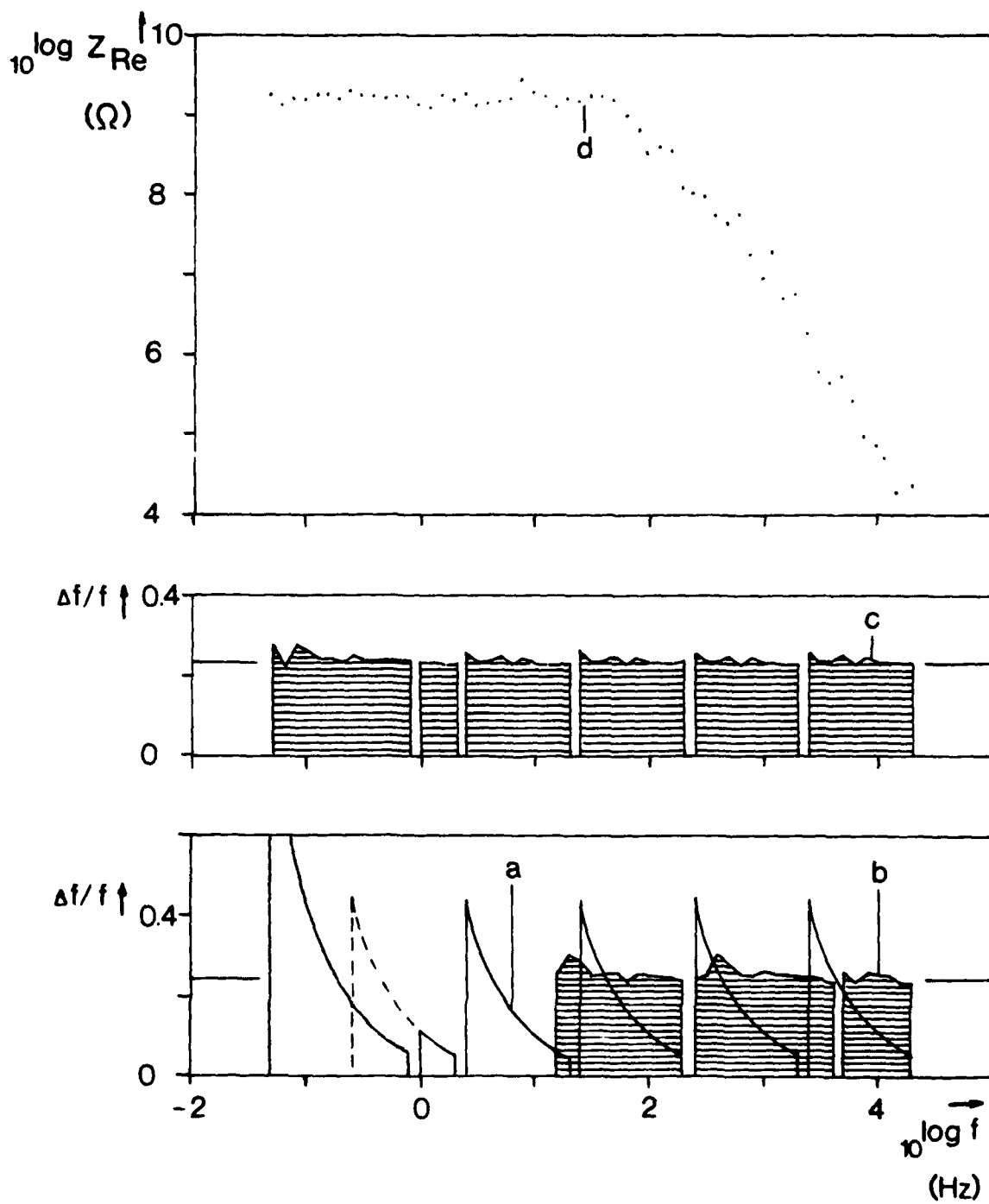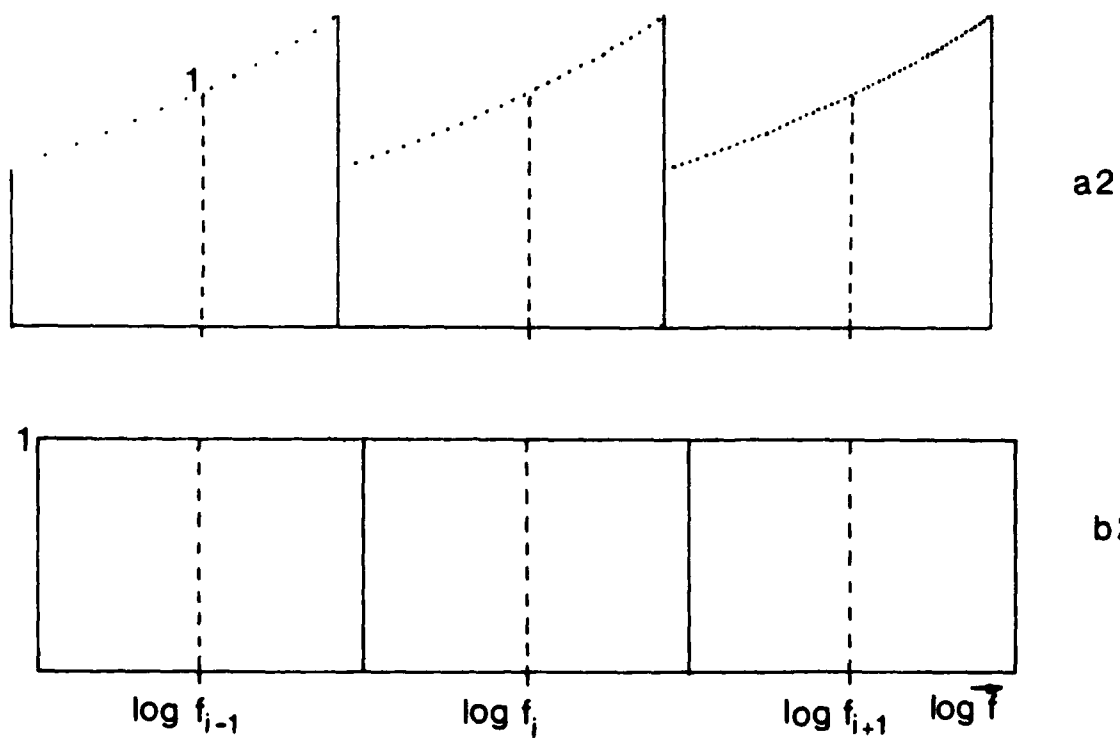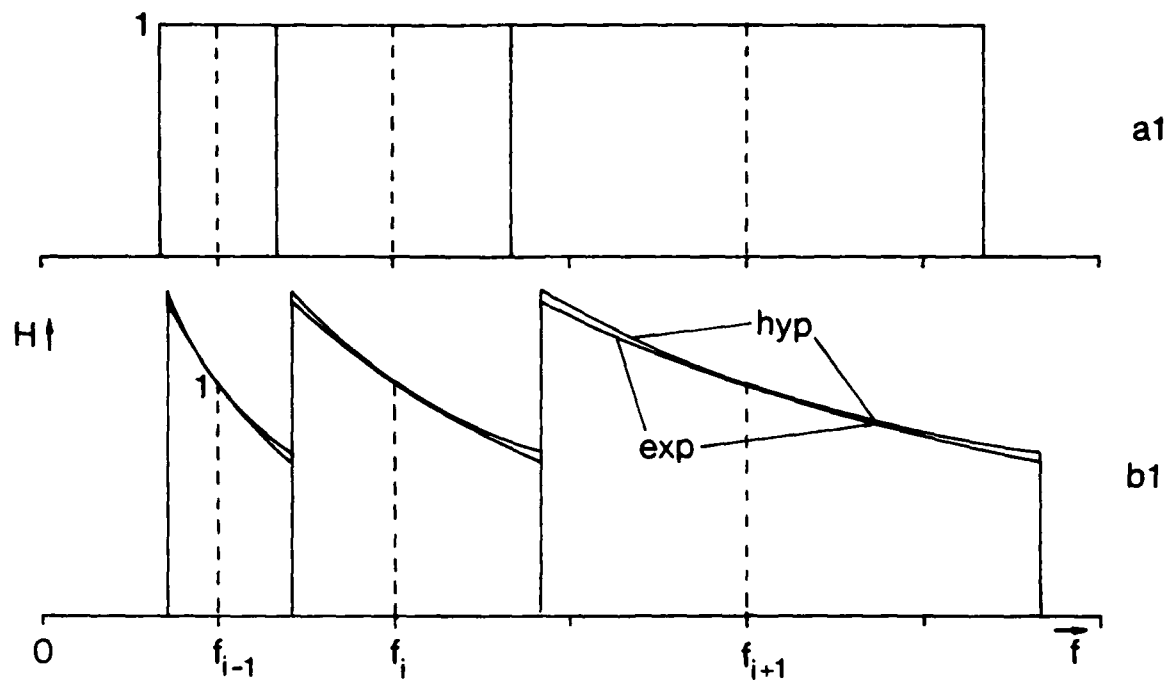(equations 16)

Figure 1.

a1

b1

H

hyp

exp

1

0 f_{i-1} f_i f_{i+1} $\vec{f}$

a2

1

b2

H^*

1

$\log f_{i-1}$ $\log f_i$ $\log f_{i+1}$ $\log \vec{f}$

Figure 2.